Diff. Eq
9th - lecture
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Physics. Dept
2nd - stage
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Differential Equation of First Order and Higher degree

The general form of diff. eq of first order and higher degree is

$$\left(\frac{dy}{dx}\right)^n + P_1\left(\frac{dy}{dx}\right)^{n-1} + P_2\left(\frac{dy}{dx}\right)^{n-2} + \cdots + P_{n-1}\frac{dy}{dx} + P_n = 0$$

where each P_i is function of x and y. If $\frac{dy}{dx} = p$, then the general form reduces to

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_{n-1} p + P_n = 0$$

Hence it also can be written as $F(x,y,p) = 0$.

we study following methods of solving diff. eq of first order and higher ~~order~~ degree.

Method of Solving diff. Eq of the form $F(x,y,p) = 0$

1. Diff. eq which are solvable for p
2. " " " " " for x
3. " " " " " for y
4. Clairaut's diff. Eq
5. Lagrange's diff. Eq

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1. Diff. Eq which are solvable for p.

suppose we can write the diff Eq
$F(x,y,p)=0$ of degree n in the form

$$(p-f_1(x,y))(p-f_2(x,y))(p-f_3(x,y)) \cdots (p-f_n(x,y))=0 \quad (*)$$

Now comparing each factor with zero we get
$p-f_i(x,y)=0$, where $i=1,2,\ldots,n$ which is
linear diff. eq.
suppose ~~function~~ solution of $p-f_i(x,y)=0$ is given
$F_i(x,y,c_i)=0$, where c_i is arbitrary constant.
The general sol. of eq $(*)$ is given by

$$F_1(x,y,c)\,F_2(x,y,c) \cdots F_n(x,y,c)=0$$

Thus, diff. eq of n degree and first order having
Linear factor $p-f_i(x,y)=0$ are known as solvable
~~of~~ for p

Example ① Solve: $xyp^3+(x^2-2y^2)p^2-2xyp=0$.
sol The given diff. eq is of degree 3 and therefor
is has three Linear factor

$$p[xyp^2+(x^2-2y^2)p-2xy]=0$$
$$p[xyp^2+x^2p-2y^2p-2xy]=0$$
$$\therefore p[(xp-2y)(yp+x)]=0$$

comparing these three Linear factor with zero, we get

① $p=0 \Rightarrow \frac{dy}{dx}=0 \Rightarrow dy=0 \Rightarrow \int dy=\int c \Rightarrow y=c$

$\boxed{p.40}$ ② $xp-2y=0 \Rightarrow x\frac{dy}{dx}=2y \Rightarrow \int\frac{dy}{y}=\int 2\frac{dx}{x} \Rightarrow \log y=\log x^2$

$\Rightarrow \log y=\log x+\log c \Rightarrow (\log y=\log xc)*c \Rightarrow y=cx^2$

(3.) $yp+x=0 \Rightarrow \frac{dy}{dx}y+x=0 \Rightarrow y\,dy+x\,dx=0$

$$\left(\frac{y^2}{2}+\frac{x^2}{2}=c\right)\times 2$$

$$y^2+x^2=c_1 \,,\quad c_1=2c$$

There for The general solution is given by multiplying these three solution of linear factors of given eq.

$$\therefore (y-c)(y-cx^2)(x^2+y^2-2c)=0$$

(2) solve; $\frac{dy}{dx}-\frac{dx}{dy}=\frac{x}{y}-\frac{y}{x}$

Sol put $p=\frac{dy}{dx} \Rightarrow \frac{dx}{dy}=\frac{1}{p}$

then $\left[p-\frac{1}{p}=\frac{x}{y}-\frac{y}{x}\right]*p$

$$p^2+p\left(\frac{y}{x}-\frac{x}{y}\right)-1=0$$

$$\left(p+\frac{y}{x}\right)\left(p-\frac{x}{y}\right)=0$$

Now comparing the Linear factors with zero, we get

(1) $p+\frac{y}{x}=0$

$\Rightarrow \left.\frac{dy}{dx}+\frac{y}{x}=0\right] \cdot \left[x\,dy+y\,dx=0\right] \div xy$

$$\int\frac{dy}{y}+\int\frac{dx}{x}=\int 0 \Rightarrow (\log y+\log x=c)^{e}$$

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$$xy=c$$

(2) $p-\frac{y}{x}=0 \Rightarrow \int y\,dy=\int x\,dx = x^2-y^2=c\,,\ c=2c_1$

Thus, the general solution can be obtained
- by multiplying the general sol. of the linear factors of given diff. Eq

$$(xy-c)(x^2-y^2-c)=0$$

which is general solution

2. Differential Eq which are solvable of y

If the differential eq of the form $F(x,y,p)=0$ can be written as $y=f(x,p)=0$, then is said to be solvable for y. In order to solve these types of diff. Eq we differentiate with respect to x we get

$$\frac{dy}{dx}=p=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial p}\frac{dp}{dx}=F\left(x,p,\frac{dp}{dx}\right) \quad (*)$$

which is in variable p and x. By eliminate p from eq (*) and $g(x,p,c)$ we get fun. $\phi(x,y,c)$ Which is general solution of the given diff. eq.
If it is not possible to eliminate p, then general sol. can be obtained by taking $x=F_1(p,c)$ and $y=F_2(p,c)$, where c is an arbitrary constant

Example: ① Solve $xp^2 - 2yp + ax = 0$

Solution: Here $y = \frac{1}{2} xp + \frac{1}{2}\frac{ax}{p}$ — ⊛, by differentiating with respect to x we get

$$\frac{dy}{dx} = \frac{1}{2}p + \frac{1}{2}x\frac{dp}{dx} + \frac{a}{2p} - \frac{ax}{2p^2}\frac{dp}{dx}$$

$\frac{1}{2}p + \frac{1}{2}\frac{a}{p} = \frac{a}{p}$

$$p = \frac{1}{2}p + \left(\frac{1}{2}x - \frac{ax}{2p^2}\right)\frac{dp}{dx} + \frac{1}{2}\frac{a}{p}$$

$$\left[p = \left(x - \frac{ax}{p^2}\right)\frac{dp}{dx} + \frac{a}{p}\right] * p^2$$

$$p^3 = p^2 x \frac{dp}{dx} - ax\frac{dp}{dx} + ap$$

$$p^3 - p^2 x\frac{dp}{dx} + ax\frac{dp}{dx} - ap = 0$$

$$(p^2 - a)\left(p - x\frac{dp}{dx}\right) = 0$$

$\therefore p - x\frac{dp}{dx} = 0$ or $p^2 - a = 0$

$\therefore x\frac{dp}{dx} = p \Rightarrow \frac{dx}{x} = \frac{dp}{p} \Rightarrow \log p = \log x + \log c$

$\therefore p = cx$

Now substitute $p = cx$ in eq (⊛)

$$y = \frac{1}{2}x(xc) + \frac{1}{2}\frac{ax}{xc}$$

$y = \frac{1}{2}cx^2 + \frac{1}{2}\frac{a}{c}$. which is a general sol

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② $xp - y + x^{\frac{3}{2}} = 0$

Sol The given eq can be express in the form $y = f(x,p)$ Therefor it is Solvable for y.

$y = xp + x^{\frac{3}{2}}$ —⊛ Differentiate with respect to x we get

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{3}{2}x^{\frac{1}{2}}$$

$$(\because \cancel{p} = \cancel{p} + x\frac{dp}{dx} + \frac{3}{2}x^{\frac{1}{2}}) \div x$$

$$\frac{dp}{dx} + \frac{3}{2}\frac{x^{\frac{1}{2}}}{x} = 0$$

$$\frac{dp}{dx} + \frac{3}{2}\frac{1}{\sqrt{x}} = 0 \Rightarrow \frac{dp}{dx} = -\frac{3}{2}\frac{1}{\sqrt{x}}$$

$$\int dp = -\int \frac{3}{2}\frac{1}{\sqrt{x}}\,dx + \int 0$$

$$\frac{3}{2}\int x^{-\frac{1}{2}}$$

$$\frac{3}{2}\cdot\frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 3\sqrt{x}$$

$p = -3\sqrt{x} + c$

$p = c - 3\sqrt{x}$

Now, substiute $p = c -$ in eq(⊛) we get

$y = x(c - 3\sqrt{x}) + x^{\frac{3}{2}}$

$y = xc - 3x \cdot x^{\frac{1}{2}} + x^{\frac{3}{2}}$

$y = xc - 3x^{\frac{3}{2}} + x^{\frac{3}{2}}$

$y = xc - 2x^{\frac{3}{2}}$

which is general Solution.

H.W : Solve $x + 2(xp - y) + p^2 = 0$

H.W EXEV $\boxed{P.44}$

Exercise-V

Solve the following differential equations.

1. $p^2-(x+3y)p+2y(x+y)=0.$ (Ans. $(y-ce^{-2x})(x+y-1-ce^{x})=0.$)
2. $p^2-7p+10.$ (Ans. $(y-5x-c)(y-2x-c)=0.$)
3. $p(p+y)=x(x+y).$ (Ans. $(2y-x^2+c)(y+x+ce^{-x}-1)=0.$)
4. $yp^2+(x-y)p-x.$ (Ans. $(x-y+c)(x^2+y^2+c)=0.$)
5. $p^3+2xp^2-y^2p^2-2xy^2p=0.$ (Ans. $(y-c)(y+x^2-c)(xy+cy+1)=0.$)
6. $p^2+2py\cot x-y^2=0.$ (Ans. $y(1\pm\cos x)=c.$)
7. $x^2p^2+xyp-6y^2=0.$ (Ans. $(y-cx^2)(x^3y-c)=0.$)
8. $y^2p^2-x^2=0.$ (Ans. $(x^2+y^2+c)(x^2-y^2+c)=0.$)
9. $p^2+2p\cos 2x-\sin^2 x=0.$ (Ans. $(2y+2x+\sin 2x+c)=0.$)