Diff. Eq.
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2nd - stage
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Nonhomogenous differential equations which can be reduced to homogenous differential equations

A differential equation of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{Ix+my+n} \quad \text{---} \quad (*)$$

is not homogenous diff. eq, but by making some change we can reduced it to the case of homogenous diff. eq.

Case-I $\boxed{\frac{a}{I} \neq \frac{b}{m}}$. In order to solve diff. eq. having this case,

Let $x = x' + h$ and $y = y' + k$, where h, k are constant.

Also, $dx = dx'$ and $dy = dy'$. Then eq (*) reduces to

$$\frac{dy'}{dx'} = \frac{ax' + by' + ah + bk + c}{Ix' + my' + Ih + mk + n} \quad \text{---} \quad (2*)$$

in this eq we select h and k by solving $ah + bk + c = 0$ and $Ih + mk + n = 0$ s.t eq (2*) will turn out to homogenous diff. eq

$$\frac{dy'}{dx'} = \frac{ax' + by'}{Ix' + my'}$$

where $al - bm \neq 0$, which is homogenous in variable x' and y'. So solve it by putting $y' = ux'$

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Case – II $\frac{a}{l} = \frac{b}{m}$. In this case $al - bm = 0$, and hence h and k will be indetermined or infinity. Hence put $\frac{a}{l} = \frac{b}{m} = t$, where t is constant in eq (*) we get

$$\frac{dy}{dx} = \frac{(lx+my)t + c}{(lx+my) + n} \quad — (3*)$$

Now by substitute $lx+my = t$ in eq(3*) we can solve the given diff. eq. ~~Lectures~~

Example ① $\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$

Solution

The diff. eq. is given by:

$$\frac{dy}{dx} = \frac{y+x-2}{y-x-4} \quad — ①$$

is not homogenous diff. eq, By comparing with eq(*)

$$\frac{dy}{dx} = \frac{ax+by+c}{lx+my+n}$$

we get $a=1, b=1, l=-1, m=1$. Here $\frac{a}{l} = -1 \neq \frac{b}{m} = 1$

Hence substitute $x = x'+h$ and $y = y'+k$ in eq(1) we get

$$\frac{dy'}{dx'} = \frac{y'+x'+(k+h-2)}{y'-x'+(k-h-4)} \quad — ②$$

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to convert eq(2) in homogenous diff. eq. we take

$$k-h-2=0, \quad k-h-4=0$$

by solving $\left.\begin{array}{l} k-h-2=0 \\ k-h-4=0 \end{array}\right\}$ بالجمع $\Rightarrow h=-1, k=3$

$+2k-6=0 \Rightarrow$

$2k=6 \Rightarrow k=3$

$h=-1$

Hence with these value of h and k eq (2) reduce to

$$\frac{dy'}{dx'} = \frac{y'+x'}{y'-x'} \quad (3)$$, which is homogenous diff. eq,

In order to solve put $y' = vx'$ and $v + x'\frac{dv}{dx'}$ in eq (3) we obtain,

$$v + x'\frac{dv}{dx} = \frac{vx' + x'}{vx' - x'} = \frac{v+1}{v-1}$$

$$x'\frac{dv}{dx} = \frac{v+1}{v-1} - v = \frac{1+2v-v^2}{v-1}$$

$$\therefore \frac{v-1}{1+2v-v^2}\,dv = \frac{dx}{x'}$$, which is seprable variable form

By integrating term both side

$$\int \frac{v-1}{1+2v-v^2}\,dv = \int \frac{dx}{x'} , \cancel{+c}\ c$$ where c is an arbitrarity constant

$$-\frac{1}{2}\int \frac{(v-1)(-2)}{(1+2v-v^2)} = \log x' + c$$

$$\left[-\frac{1}{2}\log(1+2v-v^2) = \log x' + c\right] \times 2$$

$$\log x' + \log(1+2v-v^2) = -2c'$$

$$\log x^2 + \log\left(1 + 2\left(\frac{y'}{x'}\right) - \left(\frac{y'}{x'}\right)^2\right) = -2c$$

$$\therefore \log(x'^2 + 2y'x' - y'^2) - \log x'^{2} + \log x'^2 = -2c$$

$\boxed{P.33}$

$$\therefore x'^2 + 2y'x' - y'^2 = e^{-2c} = c'$$

by substituting $\left.\begin{array}{l} x' = x+1 \\ y' = y-3 \end{array}\right\} \Rightarrow$ we get $x^2 + 2xy - 4x + 8y - 14 = \hat{c}$ which is general sol.

Example (2) $(x-y+2)\,dx+(2x-2y-4)\,dy=0$

Sol ~~The Solution~~

The diff. eq. is given by

$$\frac{dy}{dx}=-\frac{x-y+2}{2(x-y)-4} \quad \text{—— (1)}$$

is not homogenous diff. eq. By comparing with

$$\frac{dy}{dx}=\frac{ax+by+c}{Ix+my+n}$$ we get $a=-1,\ b=1,\ I=2,\ m=-2$

Here $\frac{a}{I}=-\frac{1}{2}=\frac{b}{m}$, therefor h,k can not be determined.

put $x-y=z$ and $1-\frac{dy}{dx}=\frac{dz}{dx}$ in eq (1)

we get

$$1-\frac{dz}{dx}+\frac{z+2}{2z-4}=0$$

$$\therefore \frac{dz}{dx}+\frac{3z-2}{2z-4}=0$$

$$\therefore \frac{3z-2}{2z-4}\,dz=dx$$, which is seprable variable form

In order to get solution integrate the terms seprately, we g

$$\int\frac{2z-2}{3z-2}\,dz=\int dx+c$$, where c is an arbitrary constant

$$\therefore \int\frac{2}{3}\,\frac{3z-2-4}{3z-2}\,dz=\int dx+c$$

~~P 33 A~~

$$\frac{2}{3}\int\left(1-\frac{4}{3z-2}\right)dz=x+c$$

P-34.A

$$\therefore \frac{2}{3}\left[x-y-\frac{4}{3}\log\left[(3(x-y)-2\right]\right]=3x+c'$$

where $c' = 3c$

$$\therefore x+2y+\frac{8}{3}\log\left[3(x-y)-2\right]+c'$$

Ex E II p(10)

$\boxed{P.34\text{-}B}$

$\boxed{P\text{-}37}$

Bernoulli's differential equations

A diff. eq of the form $\frac{dy}{dx} + Py = Qy^n$, $n \in \mathbb{R} \setminus \{0\}$ is said Bernoulli's diff. eq.

In order to solve Bernoulli's diff. eq. we will use the method of solving Linear diff. eq.

Bernoulli's diff. eq. is given by

$$\frac{dy}{dx} + Py = Qy^n \quad , n \in \mathbb{R} \setminus \{1\} \quad - *$$

Divide Both sides by y^n we get

$$y^{-n} \frac{dy}{dx} + P y^{1-n} = Q \quad - (2*)$$

Now multiply by $(1-n)$ both side we get

$$(1-n) y^{-n} \frac{dy}{dx} + (1-n) P y^{1-n} = (1-n) Q$$

Now put $v = y^{(1-n)}$ and $\frac{dv}{dx} = (1-n) y^{-n} \frac{dy}{dx}$ in eq (2*) we get

$$\frac{dv}{dx} + (1-n) P v = (1-n) Q$$

Which is Linear in variable v and can be solved by method of linear diff. eq. Hence substitute

$$\therefore I.F = e^{\int P dx} = e^{\int (1-n) P dx}$$

in eq $v e^{\int P dx} = \int Q e^{\int P dx} + C$

$\boxed{P\text{-}35}$

$$\therefore v e^{\int (1-n) P dx} = \int (1-n) Q e^{\int (1-n) P dx} dx + C$$

$$\therefore\ y^{1-n} e^{\int (1-n) p\,dx} = \int (1-n) Q e^{\int (1-n) p\,dx}\,dx + c$$

where c is an arbitrary constant. which is general solution

Ex ① Solve $x\frac{dy}{dx} + y = x^3 y^6$

Sol

The given diff. eq is not Linear in x also not Linear. To convert it not Bernoulli's form we divide the eq by xy^6 we get

$$y^{-6}\frac{dy}{dx} + y^{-5}\cdot\frac{1}{x} = x^2 \quad \text{— ①}$$

$\therefore$ put $y^{-5} = v$ and $-5y^{-6}\frac{dy}{dx} = \frac{dv}{dx}$ in eq (1)

we get

$$\frac{dv}{dx} - \frac{5}{x}v = -5x^2$$ which is linear in V.

Hence comparing with general of linear diff. eq

$$\frac{dy}{dx} + Py = Q$$

we get $P = -\frac{5}{x}$, $Q = -5x^2$. Now

$$I.F = e^{\int p\,dx} = e^{\int -\frac{5}{x}dx} = e^{-5\ln x} = e^{\ln x^{-5}} = x^{-5}$$

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$$v e^{\int p\,dx} = \int Q e^{\int p\,dx}\,dx + c$$, where c is an arbitrary constant

$$y^{-5}x^{-5} = \int -5x^2 x^{-5}\,dx + c$$

$$y^{-5}x^{-5} = \frac{5}{2}x^{-2} + c$$, which is general solution

② Solve $x\frac{dy}{dx} - y = y^2 \log x$

<u>Sol</u> : to convert this eq. in form of Bernoulli's diff. eq., we divide both sides by x, we get

$$\frac{dy}{dx} - \frac{y}{x} = \frac{\log x}{x} y^2$$

Now comparing with the general form of Bernoulli's diff. eq $\frac{dy}{dx} + Py = Qy^n$, we get $P = -\frac{1}{x}$

$Q = \frac{\log x}{x}$

with $n = 2$

Therefore the solution is given by

$$y^{1-n} e^{\int (1-n) P\, dx} = \int (1-n) Q\, e^{\int (1-n) P\, dx} + c$$

$$\therefore y^{-1} x = \int -\frac{\log x}{x}\, dx + c$$

$$\therefore -\int \log x\, dx + c \Rightarrow -\left[\log x - \int \frac{1}{x}\, dx\right] + c$$

$$\therefore x = y(c + x - x\log x)$$

$\boxed{p\text{-}37}$ which is a general sol. of the give diff. eq

Remark : The general form of Bernoulli's diff eq $\frac{dy}{dx} + Py = Qy^n$; $n \in \mathbb{R} \setminus \{0\}$

is given by $f'(y)\frac{dy}{dx} + f(y)P = Q$

In order to solve this we put $u = f(y)$ we get

$$\frac{du}{dx} = f'(y)\frac{dy}{dx}$$

in general form we get

$\frac{du}{dx} + Pu = Q$ which is linear diff. eq.

~~Examples~~:

H.W EXE III

$\boxed{P\text{-}38}$