physics Dept.
2nd - stage
Dr- mayadah-K

Linear differential equations

A differential equation of the form

$$\boxed{\frac{dy}{dx} + P y = Q}$$

where P and Q are constant or functions of x only, is known as a first order Linear differential eq.

Another form of first order Linear diff-eq. is

$$\boxed{\frac{dx}{dy} + P_1 X = Q_1}$$

where, P_1 and Q_1 are constants or functions of y only.

To solve the first order Linear diff. eq. of the type

$$\frac{dy}{dx} + Py = Q \quad \text{--- } (*)$$

Multiply both sides of eq (*) by a function of x say g(x) to get

$$g(x)\frac{dy}{dx} + P\cdot(g(x))y = Q\cdot g(x) \quad \text{--- } (2*)$$

$\boxed{P.22}$

derivative of R.H.S of $(y \cdot g(x))$

i.e $g(x)\frac{dy}{dx} + P \cdot g(x) y = \frac{d}{dx}[y \cdot g(x)]$

or $g(x)\frac{dy}{dx} + P \cdot g(x) \cdot y = g(x)\frac{dy}{dx} + y\, g'(x)$

$$P \cdot g(x) = g'(x)$$

or $P = \frac{g'(x)}{g(x)}$

Integrating both sides with respect to x, we get

$$\int P\,dx = \int \frac{g'(x)}{g(x)}\,dx$$

or $\int P\,dx = \log(g(x))$

$$g(x) = e^{\int P(x)}$$

on multiplying the eq (1) by $g(x) = e^{\int P(x)}$, the L.H.S becomes the derivative of some function of x and y. This fun. $g(x) = e^{\int P(x)}$ is called $\boxed{\text{Integrating Factor (I.F)}}$ of given diff-eq.

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Substituting the value of $g(x)$ in eq (2*)
we get

$$e^{\int p\,dx}\frac{dy}{dx}+P\,e^{\int p\,dx}\,y=Q\cdot e^{\int p\,dx}$$

$$\frac{d}{dx}\left(y\,e^{\int p\,dx}\right)=Q\,e^{\int p\,dx}$$

Integrating both sides with respect to x, we get

$$\left[y\cdot e^{\int p\,dx}=\int\left(Q\cdot e^{\int p\,dx}\right)dx\right]\div e^{\int p\,dx}$$

$$y=e^{-\int p\cdot dx}\cdot\int\left(Q\cdot e^{\int p\,dx}\right)dx+C$$

which is the general solution of the diff. eq.

Steps involved to solve first order linear diff. equation,

(i) write the given diff. eq in the form $\frac{dy}{dx}+Py=Q$ where P, Q are constant or fun. of x only.

(ii) Find the Integrating Factor $(I.F)=e^{\int p\,dx}$

(iii) write the solution of the given diff eq. as

$$y\,(I.F)=\int(Q\times I.F)\,dx+C$$

In case the first order linear diff. eq is in the form $\frac{dx}{dy}+P_1x=Q_1$

$\boxed{P\cdot 27}$

where, P_1 and Q_1 are constants or function of y only. Then $I.F = e^{\int P_1 dy}$ and the solution of the diff. eq is given by

$$x.(I.F) = \int (Q_1 * I.F)\,dy + C$$

Example:- Find the general solution of diff. eq.

$$\frac{dy}{dx} - y = \cos x \quad \text{—} \ (*)$$

solution: Given diff. eq is of the form

$$\frac{dy}{dx} + Py = Q \quad , \text{where } P = -1 \ \& \ Q = \cos x$$

Therefor $I.F = e^{\int P\,dx} = e^{\int -1\,dx} = e^{-x}$ — $(*)$

multiplying both sides of eq$(*)$ by I.F (e^{-x}) we get

$$\underbrace{e^{-x} \frac{dy}{dx} - e^{-x} y}_{} = e^{-x} \cos x$$

$$\frac{d}{dx}(y e^{-x}) = e^{-x} \cos x$$

on integrating both sides with respect to x, we get

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$$y e^{-x} = \int e^{-x} \cos x\,dx + C \quad \text{—} \ (*)$$

Let $I = \int e^{-x} \cos x\,dx$ uv تكامل بالتجزئة

let $u = \cos x$, $dv = e^{-x}$

$du = -\sin x$ $v = -e^{-x}$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$$\cos x\left(\frac{e^{-x}}{-1}\right) - \int \underbrace{-\sin x(-e^{-x})\,dx}$$

$u = \sin x \quad dv = e^{-x}$

$du = \cos x\,dx \quad v = \frac{e^{-x}}{-1}\,dx$

$\int u\,dv = uv - \int v\,du$

$= -\,x e^{x} - \int -e^{x}\cos x\,dx$

$$\therefore -\cos x\,e^{-x} + \sin x\,e^{-x} - \underbrace{\int \cos x\,e^{-x}\,dx}_{I}$$

$$I = -\cos x\,e^{-x} + \sin x\,e^{-x} - I$$

$$2I = -\cos x\,e^{-x} + \sin x\,e^{-x}$$

$$I = \frac{-\cos x\,e^{-x} + \sin x\,e^{-x}}{2}$$

$$I = \left(\frac{\sin x - \cos x}{2}\right)e^{-x}$$

substituting The value of I in eq (1)

$$y e^{-x} = \left(\frac{\sin x - \cos x}{2}\right)e^{-x} + C$$

$$y = \left(\frac{\sin x - \cos x}{2}\right) + Ce^{x}$$

which is general solution of the given diff. eq.

$\boxed{\text{P. 26}}$

Example: Find The general solution of diff. eq

$$x \frac{dy}{dx} + 2y = x^2 \quad (x \neq 0)$$

Sol

The given diff. eq. is

$$\left[x \frac{dy}{dx} + 2y = x^2 \right] \div x$$

we get

$$\frac{dy}{dx} + \frac{2}{x} y = x$$

which is a linear diff. eq. of type

$$\frac{dy}{dx} + P y = Q \quad , \text{where } P = \frac{2}{x}$$

$$Q = x$$

so $I.F = \int e^{p\,dx} = \int e^{\frac{2}{x} dx}$

Therefore, solution of The given eq is given by

$$y \cdot x^2 = \int (x)(x^2)\, dx + c = \int x^3\, dx + c$$

$$\left[y \cdot x^2 = \frac{x^4}{4} + c \right] \div x^2$$

$$y = \frac{x^2}{4} + c x^{-2}$$

which is general solution of the given diff. eq.

Example: Find the general solution of the diff-eq $y\,dx - (x+2y^2)dy = 0$ — (*)

sol

The given diff-eq (*) can be written as

$$\left[y\,dx = (x+2y^2)\,dy\right] \div y$$

$$\cancel{y\,dx = x\left(1+\frac{2y^2}{x}\right)dy} \div y$$

$$dx = \frac{x+2y^2}{y}\,dy$$

$$\left[dx = \left(\frac{x}{y} + \frac{2y^2}{y}\right)dy\right] \div dy$$

$$\frac{dx}{dy} = \frac{x}{y} + 2y$$

$$\frac{dx}{dy} - \frac{x}{y} = 2y$$

This is linear diff. eq. of the type

$$\frac{dx}{dy} + P_1 x = Q_1 \qquad \text{where } P_1 = -\frac{1}{y}, \quad Q_1 = 2y$$

Therefore

$$I.F = e^{\int p\,dx} = e^{\int -\frac{1}{y}dx} = e^{-\log y}$$

$$= e^{\log y^{-1}} = y^{-1} = \frac{1}{y}$$

Hence, the solution of the given diff. eq. is

$$x \cdot \frac{1}{y} = \int (2y)\left(\frac{1}{y}\right)dy + c$$

$$\frac{x}{y} = \int (2\,dy) + c$$

$$\frac{x}{y} = 2y + C$$

$$x = 2y^2 + Cy$$

which is general solution of the given diff. eq.

Example: Find the pardicular solution of the diff. eq.

$$\left(\frac{dy}{dx}\right) + y \cot x = 2x + x^2 \cot x \quad (x \neq 0)$$

given that $y = 0$ when $x = \frac{\pi}{2}$.

sol The given eq is a Linear diff. eq of the type

$$\frac{dy}{dx} + P(y) = Q \qquad \boxed{\begin{array}{l} \text{where } P = \cot x \\ Q = 2x + x^2 \cot x \end{array}}$$

$$I.F = e^{\int \cos x \, dx} = e^{\log \sin x} = \sin x$$

Hence, The solution of the diff. eq is given by

$$y\,(I.F) = \int (Q * I.f)\,dx + c$$

$$y \sin x = \int (2x + x^2 \cot x) * \sin x \; dx + c$$

$$\cot x = \left(\frac{\cos x}{\cancel{\sin x}}\right) * \cancel{\sin x}$$

$$y \cdot \sin x = \int 2x \cdot \sin x + x^2$$

$\boxed{P\text{-}29}$

$$y \sin x = \int \underline{2x \sin x}\, dx + \int x^2 \cos x \, dx + C$$

$$u = \sin x \qquad dv = 2x \, dx$$

$$du = \cos x \, dx \qquad v = \frac{2x^2}{2}$$

$$= uv - \int v \, du$$

$$\int 2x \sin x \, dx = \left(\frac{2x^2}{2}\right) \sin x - \int \frac{2x^2}{2} \cos x \, dx$$

$\therefore \Rightarrow$

$$y \sin x = \left(\frac{2x^2}{2}\right) \sin x - \cancel{\int \cos x \left(\frac{2x^2}{2}\right)} + \cancel{\int x^2 \cos x \, dx}$$

$$y \sin x = x^2 \sin x + C \quad \text{---} \quad (*)$$

substituting $y = 0$ and $x = \frac{\pi}{2}$ in eq $(*)$

we get

$$0 = \left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}\right) + c$$

$$c = \frac{-\pi^2}{4}$$

substituting The value of c in eq$(*)$, we get

$$y \sin x = x^2 \sin x - \frac{\pi^2}{4}$$

$$y = x^2 - \frac{\pi^2}{4 \sin x} \qquad (\sin x \neq 0)$$

which is particular solution of given diff. eq.

$\boxed{P.30}$