Differential equation
3-4th Lecture
13/10/2024

2nd stage
Physical Dept.
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- Homogenous differential equations

Consider the following functions in x and y

$F_1(x,y) = y^2 + 2xy$, $F_2(x,y) = 2x - 3y$

$F_3(x,y) = \cos\left(\frac{y}{x}\right)$, $F_4(x,y) = \sin x + \cos y$

If we replace x and y by λx and λy respectively in the above functions, for any nonzero constant λ, we get

$$F_1(\lambda x, \lambda y) = \lambda^2 y^2 + 2\lambda x \lambda y$$
$$= \lambda^2 (y^2 + 2xy) = \lambda^2 F_1(x,y)$$

$$F_2(\lambda x, \lambda y) = 2\lambda x - 3\lambda y$$
$$= \lambda(2x - 3y)$$
$$= \lambda F_2(x,y)$$

$$F_3(\lambda x, \lambda y) = \cos\left(\frac{\lambda y}{\lambda x}\right) = \cos\left(\frac{y}{x}\right) = \lambda^0 F_3(x,y)$$

$$F_4(\lambda x, \lambda y) = \sin(\lambda x) + \cos(\lambda y)$$
$$\neq \lambda^n F_4(x,y) \quad \text{for any } n \in \mathbb{N}$$

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Here, we observe that functions F_1, F_2, F_3 can be written in the form

$$F(\lambda x, \lambda y) = \lambda^n F(x,y),$$

but F_4 can not be written in this form.

This leads to the following definition:

A function $F(x,y)$ is said **homogenous function of degree n** if

$$F(\lambda x, \lambda y) = \lambda^n F(x,y) \quad \text{for any nonzero constant } \lambda.$$

الدالة $F(x,y)$ تسمى دالة متجانسة من الدرجة n اذا يمكن كتابتها بالصيغة التالية

$$F(\lambda x, \lambda y) = \lambda^n F(x,y)$$

- We note that in the above examples, F_1, F_2, F_3 are homogenous functions of degree $2, 1, 0$ respectively but F_4 is not homogenous function.

- We also observe that

الدوال السابقة (الموجودة بالصفحة السابقة) يمكن كتابتها ايضاً بالشكل الاتي

$$F_1(x,y) = x^2\left(\frac{y^2}{x^2} + \frac{2y}{x}\right) = x^2 h_1\left(\frac{y}{x}\right)$$

or $$F_1(x,y) = y^2\left(1 + \frac{2x}{y}\right) = y^2 h_2\left(\frac{x}{y}\right)$$

$$F_2(x,y) = x^1\left(2 - \frac{3y}{x}\right) = x^1 h_3\left(\frac{y}{x}\right)$$

or $$F_2(x,y) = y^1\left(2\frac{x}{y} - 3\right) = y^1 h_4\left(\frac{x}{y}\right)$$

$\boxed{P.11}$

$$F_3(x,y) = x^0 \cos\left(\frac{y}{x}\right) = x^0 h_5\left(\frac{y}{x}\right)$$

$$F_4(x,y) \neq x^n h_6\left(\frac{y}{x}\right), \text{ for any } n \in N$$

or $$F_4(x,y) \neq y^n h_7\left(\frac{x}{y}\right), \text{ for any } n \in N$$

- A differential equation of the form $\frac{dy}{dx} = F(x,y)$ is said to be homogenous if $F(x,y)$ is a homogenous function of degree zero.

- To solve

To solve a homogenous ~~function~~ differential equation of the type

$$\frac{dy}{dx} = F(x,y) = g\left(\frac{y}{x}\right) \quad \text{--- (1)}$$

we make the substitution

$$y = v \cdot x \quad \text{--- (2)}$$

$$v = \frac{y}{x} \quad \text{--- (3)}$$

Differentiating eq(2) with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{--- (4)}$$

Substituting the value of $\frac{dy}{dx}$ (eq (4)) in eq(1) we get

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$$v + x\frac{dv}{dx} = g(v)$$

or $$x\frac{dv}{dx} = g(v) - v \quad \text{--- (5)}$$

1)

Exampl 10: Show That The differential equation $(x-y)\frac{dy}{dx} = x+2y$ is homogenous and solve it.

Solution: The given differential equation can be expressed as

$$\frac{dy}{dx} = \frac{x+2y}{x-y}$$

Let $F(x,y) = \frac{x+2y}{x-y}$

Now $F(\lambda x, \lambda y) = \frac{\lambda x + 2\lambda y}{\lambda x - \lambda y} = \frac{\cancel{\lambda}(x+2y)}{\cancel{\lambda}(x-y)}$

$$= \lambda^0 F(x,y)$$

Therefore, $F(x,y)$ is homogenous function of degree Zero, So the given differential equation is homogenous differential

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$$\frac{dy}{dx} = \left[\frac{\cancel{x}(1+\frac{2y}{x})}{\cancel{x}(1-\frac{y}{x})}\right] = \left[\frac{1+\frac{2y}{x}}{1-\frac{y}{x}}\right] = g\left(\frac{y}{x}\right) \quad \dots(2)$$

R.H.S of diff. eq (2) is of the form

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Separating the variables in eq (5), we get

$$\frac{dv}{g(v)-v} = \frac{dx}{x} \quad \cdots (6)$$

integrating both sides of eq (6), we get

$$\int \frac{dv}{g(v)-v} = \int \frac{1}{x}\, dx + C \quad \cdots (7)$$

eq (7) gives general solution of the diff. eq. when we replace v by $\frac{y}{x}$

Note If the homogenous diff. eq is in the form $\frac{dx}{dy} = F(x,y)$, where $F(x,y)$ is homogenous fn. of degree zero. Then we make substitution

$$\frac{x}{y} = v \Rightarrow x = vy, \quad v = \frac{x}{y}$$

and we proced furthe to find the general solution as discussed above by written

$$\frac{dx}{dy} = F(x,y) = h\left(\frac{x}{y}\right)$$

Example 10: show that the differe. eq.
$(x-y)\frac{dy}{dx} = x+2y$ is homogenous
and solve it.

Solution: The given diff. eq. can be expressed as

$$\frac{dy}{dx} = \frac{x+2y}{x-y} \quad \dots (1)$$

Let $F(x,y) = \frac{x+2y}{x-y}$.

Now $F(\lambda x, \lambda y) = \frac{\lambda x + 2\lambda y}{\lambda x - \lambda y} = \frac{\cancel{\lambda}(x+2y)}{\cancel{\lambda}(x-y)}$

$= \lambda^0 \cdot f(x,y)$

Therefor The $F(x,y)$ is homogenous fun. of degree Zero.

Let $\frac{dy}{dx} = \frac{\cancel{x}(1+\frac{2y}{x})}{\cancel{x}(1-\frac{y}{x})} = \left(\frac{1+\frac{2y}{x}}{1-\frac{y}{x}}\right) = g\left(\frac{y}{x}\right) \quad \dots (2)$

To Solve it we make The substitution

$$y = vx \quad \dots (3)$$

$$v = \frac{y}{x} \quad \dots (4)$$

Differentiating eq (3) with respect to x we get

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$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \dots (5)$$

substituting The value of y (eq (3)) and $\frac{dy}{dx}$ (eq (5)) and v in eq (2)

$$v + x\frac{dy}{dx} = \frac{1+2v}{1-v}$$

$$x\frac{dy}{dx} = \frac{1+2v}{1-v} - v$$

$$x\frac{dy}{dx} = \frac{v^2+v+1}{1-v}$$

$$\frac{v-1}{v^2+v+1}\,dv = \frac{-dx}{x} \quad \text{---} \ (6)$$

integrating both sides of eq (6), we get

$$\int \frac{v-1}{v^2+v+1}\,dv = -\int \frac{dx}{x}$$

$$\frac{1}{2}\int \frac{(v-1)\cdot 2}{v^2+v+1}\,dv = -\log|x| + C_1$$

$$\frac{1}{2}\int \frac{2v-2}{v^2+v+1}\,dv = -\log|x|$$

$$\frac{1}{2}\int \frac{2v+1-3}{v^2+v+1}\,dv = -\log|x|$$

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$$\frac{1}{2}\int \frac{2v+1}{v^2+v+1}\,dv - \frac{3}{2}\int \frac{1}{v^2+v+1}\,dv = -\log|x| + c$$

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$$\frac{1}{2}\log|v^2+v+1| - \frac{3}{2}\int \frac{1}{v^2+v+1}\,dv = -\log|x| + c$$

$$\frac{1}{2}\log|v^2+v+1| - \frac{3}{2}\int \frac{1}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dv = -\log|x| + c$$

$$\cancel{\frac{1}{2}\log|v^2+v+1| - \frac{3}{2}\times\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2v+1}{}\right)}$$

$$\frac{1}{2}\log|v^2+v+1| - \frac{3}{2}\int \frac{1}{\frac{\sqrt{3}}{2}\left(1+\frac{(v+\frac{1}{2})^2}{(\frac{\sqrt{3}}{2})^2}\right)} = -\log|x|+c$$

$$\frac{\frac{2v+1}{\cancel{2}}}{\frac{\sqrt{3}}{\cancel{2}}} = \frac{2v+1}{\sqrt{3}}$$

$\sqrt{3}\cdot\cancel{\sqrt{3}}$

$$\frac{1}{2}\log|v^2+v+1| - \frac{3}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{\sqrt{3}}}\int \frac{1}{1+\left(\frac{2v+1}{\sqrt{3}}\right)^2} = -\log|x|+c$$

$$\frac{1}{2}\log|v^2+v+1| - \sqrt{3}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) = -\log|x|+c_1$$

$$\frac{1}{2}\log|v^2+v+1| + \log|x| = \sqrt{3}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right)+c_1 \quad \text{--- } (7)$$

Replacing v by $\frac{y}{x}$ in eq (7)

$(|x| = \sqrt{x^2} \Rightarrow \log(x^2)^{\frac{1}{2}} = \frac{1}{2}\log(x^2))$

$$\frac{1}{2}\log\left|\left(\frac{y}{x}\right)^2+\frac{y}{x}+1\right| + \frac{1}{2}\log x^2 = \sqrt{3}\tan^{-1}\left(\frac{2(\frac{y}{x})+1}{\sqrt{3}}\right)+c_1$$

$*(\log x + \log y = \log(x\cdot y))$

$$\left[\frac{1}{2}\log\left|\left(\frac{y^2}{x^2}+\frac{y}{x}+1\right)x^2\right| = \sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}\,x}\right)+c_1\right]$$

$$\log|y^2+yx+x^2| = 2\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}\,x}\right)+c$$

$$\log|x^2+yx+y^2| = 2\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}\,x}\right)+c$$

which is general solution of the diff. eq (1)

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Example 11 Show that the diff. eq

$$x \cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$$

is homogenous and solve it

<u>Solution</u> The given diff. eq. can be written as

$$\frac{dy}{dx} = \frac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)} \quad \cdots (1)$$

$$F(x,y) = \frac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)}$$

$$F(\lambda x, \lambda y) = \frac{\lambda y \cos\left(\frac{\lambda y}{\lambda x}\right) + \lambda x}{\lambda x \cos\left(\frac{\lambda y}{\lambda x}\right)}$$

$$= \frac{\cancel{\lambda}\left(y\cos\left(\frac{y}{x}\right) + x\right)}{\cancel{\lambda}\left(x\cos\left(\frac{y}{x}\right)\right)} = \lambda^{0} F(x,y)$$

Thus $F(x,y)$ is homogenous fun of degree zero

To solve it, make the substitution

$$y = vx \quad \cdots (2)$$

$$v = \frac{y}{x} \quad \cdots (3)$$

differentiating eq (2) with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \cdots (4)$$

Substituting values of y (eq ②), v (eq (3)) & $\frac{dy}{dx}$ (eq (4)) in eq (1), we get
:

$$v + x\frac{dv}{dx} = \frac{vx\cos v + x}{x\cos v}$$

$$v + x\frac{dv}{dx} = \frac{\cancel{x}(v\cos v + 1)}{\cancel{x}\cos v}$$

$$x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v} - v$$

$$x\frac{dv}{dx} = \frac{v\cancel{\cos v} + 1 - v\cancel{\cos v}}{\cos v}$$

$$x\frac{dv}{dx} = \frac{1}{\cos v}$$

$$[x\cos v\, dv = dx] \div x$$

$$\cos v\, dv = \frac{dx}{x}$$

$$\int \cos v\, dv = \int \frac{dx}{x}$$

$$\sin v = \log|x| + \log|C| \qquad (\log x + \log y = \log xy)$$

$$\sin v = \log|Cx| \quad \text{--- ⑦}$$

Replacing v by $\frac{y}{x}$ in eq (7) we get

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$$\sin\left(\frac{y}{x}\right) = \log|Cx|$$

which is general solution of diff. eq (1)

Example 12 show The diff. eq

$$2y e^{\frac{x}{y}} dx + \left(y - 2x e^{\frac{x}{y}} \right) dy = 0$$

is homogenous and find it particualr solution given that $x=0$, when $y=1$.

Solution

The given diff. eq. can be written as

$$2y e^{\frac{x}{y}} dx = 2x e^{\frac{x}{y}} - y \; dy$$

هنا كتبنا $\frac{dx}{dy}$ وليس $\frac{dy}{dx}$ لان لدينا الدالة نفسها $\frac{x}{y}$

$$\frac{dx}{dy} = \frac{2x e^{\frac{x}{y}} - y}{2y e^{\frac{x}{y}}} \quad \dots \quad (1)$$

$$\therefore F(x,y) = \frac{2x e^{\frac{x}{y}} - y}{2y e^{\frac{x}{y}}}$$

$$F(\lambda x, \lambda y) = \frac{2\lambda x \, e^{\frac{\lambda x}{\lambda y}} - \lambda y}{2\lambda y \, e^{\frac{\lambda x}{\lambda y}}}$$

$$= \frac{\cancel{\lambda}(2x e^{\frac{x}{y}} - y)}{\cancel{\lambda}(2y e^{x/y})} = \lambda^{0} F(x,y)$$

Thus $F(x,y)$ is homogenous function of degree Zero.

To solve it, we make The substitution

فرضنا الفرضية دي "$x=vy$" وليس ($y=vx$)

$$x = vy \quad \dots \quad (2)$$

$$v = \frac{x}{y} \quad \dots \quad (3)$$

Differentiating eq (2) with respet to y, we get

$$\frac{dx}{dy} = v + y \frac{dv}{dx} \quad \dots \quad (4)$$

substituting the value x (eq (2) , v (eq (3) &
and $\frac{dx}{dy}$ (eq (4)) in eq (1) , we get

$$v + y\frac{dv}{dy} = \frac{2vye^{v} - y}{2ye^{v}}$$

$$v + y\frac{dv}{dy} = \frac{\cancel{y}(2ve^{v} - 1)}{\cancel{y}(2e^{v})}$$

$$v + y\frac{dv}{dy} = \frac{2ve^{v} - 1}{2e^{v}}$$

$$y\frac{dv}{dy} = \frac{2ve^{v} - 1}{2e^{v}} - v$$

$$y\frac{dv}{dy} = \frac{\cancel{2ve^{v}} - 1 - \cancel{2ve^{v}}}{2e^{v}}$$

$$y\frac{dv}{dy} = \frac{-1}{2e^{v}}$$

$$\left[2ye^{v}\,dv = -dy\right] \div y$$

$$2e^{v}\,dv = -\frac{dy}{y}$$

$$\int 2e^{v}\,dv = -\int \frac{dy}{y}$$

$$2e^{v} = -\log|y| + C \quad \ldots (5)$$

replacing $v = \frac{x}{y}$, we get

p.2c

$$2e^{\frac{x}{y}} = -\log|y| + C$$

substituting x=0 and y=1 in eq (5) we get

$$2e^{\frac{0}{1}} = -\log|1| + C \Rightarrow 2(1) = 0 + C \Rightarrow \boxed{C = 2}$$

Substituting $c=2$ in eq (5)

$$2e^{\frac{x}{y}} + \log|y| = c$$

which is particular solution for diff. eq^n

H.w EXE 9.4

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