$$\therefore \log(x'^2 + 2x'y' - y'^2) - \log x'^2 + \log x'^2 = -2c$$
$$\therefore x'^2 + 2x'y' - y'^2 = e^{-2c} = c'$$

by substituting $x' = x+1$ and $y' = y-3$, we get $x^2+2xy-y^2-4x+8y-14 = c'$, which is general equation of given differential equation. (2)$(x-y+2)dx + (2x-2y-4)dy = 0$
Solution: The differential equation is given by,

$$\frac{dy}{dx} = -\frac{x-y+2}{2(x-y)-4} \qquad (2.12)$$

is not homogeneous differential equation. By comparing with (2.6) we get $a=-1, b=l, l=2, m=-2$. Here, $\frac{a}{l} = -\frac{1}{2} = \frac{b}{m}$. Therefore h and k can not be determined. Put $x-y=z$ and $1-\frac{dy}{dx} = \frac{dz}{dx}$ in equation (2.12) we get,

$$1 - \frac{dz}{dx} + \frac{z+2}{2z-4} = 0$$
$$\therefore \frac{dz}{dx} + \frac{3z-2}{2z-4} = 0$$
$$\therefore \frac{2z-4}{3z-2}dz = dx, \text{ which is separable variable form.}$$

In order to get solution integrate the terms separately we get

$$\int \frac{2z-4}{3z-2}dz = \int dx + c, \text{ where c is an arbitrary constant}$$
$$\therefore \int \frac{2}{3}\frac{3z-2-4}{3z-2}dz = \int dx + c$$
$$\therefore \frac{2}{3}\int\left(1-\frac{4}{3z-2}\right)dz = x+c$$
$$\therefore \frac{2}{3}\left[x-y-\frac{4}{3}\log[3(x-y)-2]\right] = 3x+c', \text{ where } c' = 3c$$
$$\therefore x+2y+\frac{8}{3}\log[3(x-y)-2]+c', \text{ which is a general solution.}$$

Exercise-II

Identify type of the following differential equations and solve them.

1. $2y\frac{dy}{dx} = x^2 + \sin 3x$. (Ans: $3y^2 = x^3 - \cos 3x + c$.)
2. $3e^x \tan y dx + (1-e^x)\sec^2 y dy = 0$. (Ans: $\tan y = c(1-e^x)^3$.)
3. $\frac{y}{x}\frac{dy}{dx} + \frac{2(x^2+y^2)-1}{x^2+y^2+1} = 0$. (Ans: $2x^2+y^2+3\log(x^2+y^2-2) = c$.)
4. $x^4\frac{dy}{dx} + x^3y + cosec(xy) = 0$. (Ans: $\cos xy + \frac{1}{2x^2} = c$.)

5. $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$. (Ans: $(x+a)(1-ay) = cy$.)

6. $x\frac{dy}{dx} = y + \cos^2\left(\frac{y}{x}\right)$. (Ans: $\tan\left(\frac{y}{x}\right) = \log|cx|$

7. $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$. (Ans: $y = x\log y + cx$).

8. $y - x\frac{dy}{dx} = \sqrt{y^2 - x^2}$. (Ans: $y + \sqrt{y^2 - x^2} = c$.)

9. $\frac{x+y+1}{x-y+1}$. (Ans: $\tan^{-1}\frac{y}{x+1} = \log\left(c\sqrt{(x+1)^2 + y^2}\right)$).

10. $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$. (Ans: $(x+y-2)(x-y)^{-3} = c$).

11. $(3y+2x+4)dx - (4x+6y+5)dy = 0$. (Ans: $21x - 42y + 9\log(14x+21y+22) = c'$).

12. $(2x+9y-20)dx = (6x+2y-10)dy$. (Ans: $(y-2x)^2 = c(x+2y-5)$).

2.4 Linear differential equations.

Definition 2.6. *A differential equation of the form* $\frac{dy}{dx} + Py = Q$, *where* P *and* Q *are either constants or functions of* x *is said to be* linear differential equation of first order. *For example,* $\frac{dy}{dx} + (\sec^2 x)y = \sec^2 x \tan x$ *is linear differential equation of first order.*

In order to solve the linear differential equation we use the method of separable variable. Linear differential equation of first order is given by

$$\frac{dy}{dx} + Py = Q, \text{ where P and Q are either constants or functions of } x. \tag{2.13}$$

First we solve $\frac{dy}{dx} + Py = 0$ by using separable variable method. For

$$\int \frac{dy}{y} = -\int Pdx + c. \text{ where } c \text{ is an arbitrary constant.}$$

$$\log y = -\int Pdx + c'.$$

$$\therefore y = e^{-\int Pdx}e^{-c'}.$$

$$\therefore y = e^{-\int Pdx}c.$$

Now differentiate on both sides with respect to x we get,

$$e^{\int Pdx}\frac{dy}{dx} + ye^{\int Pdx}P = 0.$$

$$e^{\int Pdx}\left(\frac{dy}{dx} + Py\right) = 0.$$

$$\therefore y^{-1}x = \int -1\frac{\log x}{x}x\,dx + c.$$

$$\therefore -\int \log x\,dx + c \implies -[\log x x - \int \frac{1}{x}x\,dx] + c.$$

$\therefore x = y(c + x - x\log x)$. *Which is a general solution of the given differential equation.*

Remark 2.11. *The general form of Bernoulli's differential equation* $\frac{dy}{dx} + Py = Qy^n;\ n \in \mathbb{R}\setminus\{0\}$ *is given by*

$$f'(y)\frac{dy}{dx} + f(y)P = Q.$$

In order to solve this we put $u = f(y)$ *we get* $\frac{du}{dx} = f'(y)\frac{dy}{dx}$ *in general form we get* $\frac{du}{dx} + Pu = Q$, *which is linear differential equation. Let us see the following examples to understand.*

Examples 2.12. *(1) Solve:* $\sin y\frac{dy}{dx} + x\cos y = x$.
Solution: Here $u = \cos y$ *and* $\frac{du}{dx} = -\sin y\frac{dy}{dx}$. *Substitute these values in given differential equation we get*

$$\frac{du}{dx} - xu = -x.$$

Which is linear differential equation in variable v. Therefore solution is given by

$$u(I.F.) = \int Q(I.F.)dx + c.$$

$$ue^{\frac{-x^2}{2}} = \int (-x)e^{\frac{-x^2}{2}}dx + c.$$

$$\cos y = \frac{1}{2} + ce^{\frac{x^2}{2}}.$$ *Which is a general solution.*

(2) Solve: $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x}(\log y)^2$.
Solution: Divide both sides by y we get

$$\frac{1}{y}\frac{dy}{dx} + \frac{1}{x}\log y = \frac{1}{x}(\log y)^2.$$

Now put $u = \log y$, *we get* $\frac{1}{y}\frac{dy}{dx} = \frac{du}{dx}$. *Substitute these values in above equation we get*

$$\frac{du}{dx} + \frac{u}{x} = \frac{u^2}{x} \implies \frac{1}{u^2}\frac{du}{dx} + \frac{1}{x}\frac{1}{u} = \frac{1}{x}$$

. *Which is in the form of Bernoulli's differential equation. By putting* $\frac{1}{u} = t$ *and solving it we get* $(\log y)^{-1} = 1 + cx$ *which is general solution of given differential equation.*

Exercise-III

Identify type of the following differential equations and solve them.

1. $\frac{dy}{dx} + y\cos x = \sin x\cos x$ (Ans: $y = \sin x + ce^{-\sin x} - 1$.)

2. $\frac{dy}{dx} + 2xy = 2x$, also $y = 3$ when $x = 0$ obtain a particular solution. (Ans: $y = 1 + ce^{-x^2}$ and P.S. is $y = 1 + 2e^{-x^2}$.)

3. $\frac{dy}{dx} + y\tan x = \sec x$. (Ans: $y = \sin x + c\cos x$.)

4. $\cos^2 x\frac{dy}{dx} + y = \tan x$. (Ans: $y = \tan x - 1 + ce^{-\tan x}$.)

5. $(1 + x^2)dy = (\tan^{-1} x - y)dx$. (Ans: $y = \tan^{-1} x - 1 + ce^{-\tan^{-1} x}$.)

6. $x\frac{dy}{dx} + 2y = x^2\log x$. (Ans: $y = \frac{x^2}{4}\log x - \frac{x^2}{16} + cx^{-2}$.)

7. $\frac{dy}{dx} + y\cot x = 5e^{\cos x}$. (Ans: $y\sin x = -5e^{\cos x + c}$.)

8. $\frac{dy}{dx} + 2y\tan x = \sin x$, also obtain particular solution with $y = 0$ when $x = \frac{\pi}{3}$. (Ans: $y\sec^2 x = \sec x + c$; P.S $= y\sec^2 x = \sec x - 2$)

9. $(x + 2y^3)\frac{dy}{dx} = y$. (Ans: $x = y^3 + cy$.)

10. $x\log x\frac{dy}{dx} + y = 2\log x$. (Ans: $y\log x = (\log x)^2 + c$.)

11. $\frac{dy}{dx} + y\tan x = y^3\sec x$. (Ans: $\cos^2 x = y^2(c + 2\sin x)$)

12. $xy(1 + xy^2)\frac{dy}{dx} = 1$. (Ans: $\frac{1}{x} = (2 - y^2) + ce^{\frac{-y^2}{2}}$.)

13. $\frac{dy}{dx} + y\tan x = \frac{\cos x}{y}$. (Ans: $y^2 = \cos^2 x\left[c + \log\tan\left(\frac{x}{4} + \frac{x}{2}\right)\right]$.)

14. $\sec^2 y\frac{dy}{dx} + x\tan y = x^3$. (Ans: $\tan y = x^3 - 3x^2 + 6x - 6 + ce^{-x}$.)

15. $(x^3y^3 + xy)dx = dy$. (Ans: $y^{-1} = 2 - x^2 + ce^{\frac{-x^2}{2}}$.)

16. $\frac{dy}{dx} + y\cos x = y^3\sin 2x$. (Ans: $y^{-2} = 2\sin x + 1 + ce^{2\sin x}$.)

17. $x\frac{dy}{dx} = y - \sqrt{y}$. (Ans: $4c^2x = (y - 1 - c^2x)^2$.)

18. $x^3\frac{dy}{dx} - x^2y + y^4 = 0$. (Ans: $y^3(3x + c) = x^3$.)

19. $\frac{dy}{dx} + y\log y = xye^x$. (Ans: $x\log y = (x - 1)e^x + c$.)