Table 5.2.1

DIFFERENTIATION FORMULA	INTEGRATION FORMULA
1. $\frac{d}{dx}[x] = 1$	$\int dx = x + C$
2. $\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r \quad (r \neq -1)$	$\int x^r \, dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$
3. $\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x \, dx = \sin x + C$
4. $\frac{d}{dx}[-\cos x] = \sin x$	$\int \sin x \, dx = -\cos x + C$
5. $\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x \, dx = \tan x + C$
6. $\frac{d}{dx}[-\cot x] = \csc^2 x$	$\int \csc^2 x \, dx = -\cot x + C$
7. $\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x \, dx = \sec x + C$
8. $\frac{d}{dx}[-\csc x] = \csc x \cot x$	$\int \csc x \cot x \, dx = -\csc x + C$

In the case where n is odd, the exponent can be reduced to 1, leaving us with the problem of integrating $\tan x$ or $\sec x$. These integrals are given by

$$\int \tan x\,dx = \ln|\sec x| + C \qquad (21)$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C \qquad (22)$$

Formula (21) can be obtained by writing

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

$$= -\ln|\cos x| + C \qquad \boxed{\begin{aligned} u &= \cos x \\ du &= -\sin x\,dx \end{aligned}}$$

$$= \ln|\sec x| + C \qquad \boxed{\ln|\cos x| = -\ln\frac{1}{|\cos x|}}$$

Formula (22) requires a trick. We write

$$\int \sec x\,dx = \int \sec x\left(\frac{\sec x + \tan x}{\sec x + \tan x}\right)dx = \int \frac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}\,dx$$

$$= \ln|\sec x + \tan x| + C \qquad \boxed{\begin{aligned} u &= \sec x + \tan x \\ du &= (\sec^2 x + \sec x\tan x)\,dx \end{aligned}}$$

The following basic integrals occur frequently and are worth noting:

$$\int \tan^2 x\,dx = \tan x - x + C \qquad (23$$

$$\int \sec^2 x\,dx = \tan x + C \qquad (24$$

Formula (24) is already known to us, since the derivative of $\tan x$ is $\sec^2 x$. Formula (2 can be obtained by applying reduction formula (19) with $n = 2$ (verify) or, alternativel by using the identity

$$1 + \tan^2 x = \sec^2 x$$

to write

$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C$$

The formulas

$$\int \tan^3 x\,dx = \tfrac{1}{2}\tan^2 x - \ln|\sec x| + C \qquad ($$

$$\int \sec^3 x\,dx = \tfrac{1}{2}\sec x\tan x + \tfrac{1}{2}\ln|\sec x + \tan x| + C \qquad ($$

can be deduced from (21), (22), and reduction formulas (19) and (20) as follows:

$$\int \tan^3 x\,dx = \frac{1}{2}\tan^2 x - \int \tan x\,dx = \frac{1}{2}\tan^2 x - \ln|\sec x| + C$$

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\int \sec x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x|$$

DUCTS OF ECANTS

If m and n are positive integers, then the integral

$$\int \tan^m x\sec^n x\,dx$$

can be evaluated by one of the three procedures stated in Table 8.3.2, depending on wh m and n are odd or even.

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$(x-y)^n = x^n - nx^{n-1}y + \dfrac{n(n-1)}{1\cdot 2}x^{n-2}y^2 - \dfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}y^3 + \cdots \pm nxy^{n-1} \mp y^n$

TABLE OF INTEGRALS

BASIC FUNCTIONS

1. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C$

2. $\displaystyle\int \frac{du}{u} = \ln|u| + C$

3. $\displaystyle\int e^u\,du = e^u + C$

4. $\displaystyle\int \sin u\,du = -\cos u + C$

5. $\displaystyle\int \cos u\,du = \sin u + C$

6. $\displaystyle\int \tan u\,du = \ln|\sec u| + C$

7. $\displaystyle\int \sin^{-1} u\,du = u\sin^{-1} u + \sqrt{1-u^2} + C$

8. $\displaystyle\int \cos^{-1} u\,du = u\cos^{-1} u - \sqrt{1-u^2} + C$

9. $\displaystyle\int \tan^{-1} u\,du = u\tan^{-1} u - \ln\sqrt{1+u^2} + C$

10. $\displaystyle\int a^u\,du = \frac{a^u}{\ln a} + C$

11. $\displaystyle\int \ln u\,du = u\ln u - u + C$

12. $\displaystyle\int \cot u\,du = \ln|\sin u| + C$

13. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C = \ln\left|\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}u\right)\right| + C$

14. $\displaystyle\int \csc u\,du = \ln|\csc u - \cot u| + C = \ln\left|\tan \tfrac{1}{2}u\right| + C$

15. $\displaystyle\int \cot^{-1} u\,du = u\cot^{-1} u + \ln\sqrt{1+u^2} + C$

16. $\displaystyle\int \sec^{-1} u\,du = u\sec^{-1} u - \ln|u + \sqrt{u^2-1}| + C$

17. $\displaystyle\int \csc^{-1} u\,du = u\csc^{-1} u + \ln|u + \sqrt{u^2-1}| + C$

POWERS OF u MULTIPLYING OR DIVIDING BASIC FUNCTIONS

44. $\int u \sin u \, du = \sin u - u \cos u + C$ 51.

45. $\int u \cos u \, du = \cos u + u \sin u + C$ 52.

46. $\int u^2 \sin u \, du = 2u \sin u + (2 - u^2) \cos u + C$ 53.

47. $\int u^2 \cos u \, du = 2u \cos u + (u^2 - 2) \sin u + C$ 54.

48. $\int u^n \sin u \, du = -u^n \cos u + n \int u^{n-1} \cos u \, du$ 55.

49. $\int u^n \cos u \, du = u^n \sin u - n \int u^{n-1} \sin u \, du$ 56.

50. $\int u^n \ln u \, du = \frac{u^{n+1}}{(n+1)^2} [(n+1) \ln u - 1] + C$

7.8.3 THEOREM.

$$\frac{d}{dx}[\sinh u] = \cosh u \frac{du}{dx} \qquad \int \cosh u\, du = \sinh u + C$$

$$\frac{d}{dx}[\cosh u] = \sinh u \frac{du}{dx} \qquad \int \sinh u\, du = \cosh u + C$$

$$\frac{d}{dx}[\tanh u] = \operatorname{sech}^2 u \frac{du}{dx} \qquad \int \operatorname{sech}^2 u\, du = \tanh u + C$$

$$\frac{d}{dx}[\coth u] = -\operatorname{csch}^2 u \frac{du}{dx} \qquad \int \operatorname{csch}^2 u\, du = -\coth u + C$$

$$\frac{d}{dx}[\operatorname{sech} u] = -\operatorname{sech} u \tanh u \frac{du}{dx} \qquad \int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$

$$\frac{d}{dx}[\operatorname{csch} u] = -\operatorname{csch} u \coth u \frac{du}{dx} \qquad \int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

7.8.5 THEOREM.

$$\frac{d}{dx}(\sinh^{-1} u) = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \quad u > 1$$

$$\frac{d}{dx}(\tanh^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| < 1$$

$$\frac{d}{dx}(\coth^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| > 1$$

$$\frac{d}{dx}(\operatorname{sech}^{-1} u) = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \quad 0 < u < 1$$

$$\frac{d}{dx}(\operatorname{csch}^{-1} u) = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}, \quad u \neq 0$$

7.8.6 THEOREM. *If $a > 0$, then*

$$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C \quad or \quad \ln(u + \sqrt{u^2+a^2}) + C$$

$$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C \quad or \quad \ln(u + \sqrt{u^2-a^2}) + C, \quad u > a$$

$$\int \frac{du}{a^2-u^2} = \begin{cases} \dfrac{1}{a}\tanh^{-1}\left(\dfrac{u}{a}\right) + C, & |u| < a \\ \dfrac{1}{a}\coth^{-1}\left(\dfrac{u}{a}\right) + C, & |u| > a \end{cases} \quad or \quad \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C, \quad |u| \neq a$$

$$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left|\frac{u}{a}\right| + C \quad or \quad -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-u^2}}{|u|}\right) + C, \quad 0 < |u| < a$$

$$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C \quad or \quad -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2+u^2}}{|u|}\right) + C, \quad u \neq 0$$

7.8.2 THEOREM.

$$\cosh x + \sinh x = e^x$$
$$\cosh x - \sinh x = e^{-x}$$
$$\cosh^2 x - \sinh^2 x = 1$$
$$1 - \tanh^2 x = \operatorname{sech}^2 x$$
$$\coth^2 x - 1 = \operatorname{csch}^2 x$$
$$\cosh(-x) = \cosh x$$
$$\sinh(-x) = -\sinh x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$
$$\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$$
$$\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$$
$$\sinh 2x = 2 \sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\cosh 2x = 2 \sinh^2 x + 1$$
$$\cosh 2x = 2 \cosh^2 x - 1$$

PYTHAGOREAN IDENTITIES

$\sin^2\theta + \cos^2\theta = 1$ $\quad$ $\tan^2\theta + 1 = \sec^2\theta$ $\quad$ $1 + \cot^2\theta = \csc^2\theta$

SIGN IDENTITIES

$\sin(-\theta) = -\sin\theta$ $\quad$ $\cos(-\theta) = \cos\theta$ $\quad$ $\tan(-\theta) = -\tan\theta$

$\csc(-\theta) = -\csc\theta$ $\quad$ $\sec(-\theta) = \sec\theta$ $\quad$ $\cot(-\theta) = -\cot\theta$

SUPPLEMENT IDENTITIES

$\sin(\pi - \theta) = \sin\theta$ $\quad$ $\cos(\pi - \theta) = -\cos\theta$ $\quad$ $\tan(\pi - \theta) = -\tan\theta$

$\csc(\pi - \theta) = \csc\theta$ $\quad$ $\sec(\pi - \theta) = -\sec\theta$ $\quad$ $\cot(\pi - \theta) = -\cot\theta$

$\sin(\pi + \theta) = -\sin\theta$ $\quad$ $\cos(\pi + \theta) = -\cos\theta$ $\quad$ $\tan(\pi + \theta) = \tan\theta$

$\csc(\pi + \theta) = -\csc\theta$ $\quad$ $\sec(\pi + \theta) = -\sec\theta$ $\quad$ $\cot(\pi + \theta) = \cot\theta$

COMPLEMENT IDENTITIES

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ $\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$

$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$ $\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$ $\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$

ADDITION FORMULAS

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

DOUBLE-ANGLE FORMULAS

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$\cos 2\alpha = 2\cos^2\alpha - 1$

$\cos 2\alpha = 1 - 2\sin^2\alpha$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

HALF-ANGLE FORMULAS

$$\sin^2\frac{a}{2} = \frac{1 - \cos\alpha}{2} \qquad \cos^2\frac{\alpha}{2} = \frac{1 + \cos\alpha}{2}$$

7.8.1 DEFINITION.

Hyperbolic sine $$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic cosine $$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic tangent $$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent $$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Hyperbolic secant $$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cosecant $$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

7.8.3 THEOREM.

$$\frac{d}{dx}[\sinh u] = \cosh u \frac{du}{dx} \qquad \int \cosh u \, du = \sinh u + C$$

$$\frac{d}{dx}[\cosh u] = \sinh u \frac{du}{dx} \qquad \int \sinh u \, du = \cosh u + C$$

$$\frac{d}{dx}[\tanh u] = \operatorname{sech}^2 u \frac{du}{dx} \qquad \int \operatorname{sech}^2 u \, du = \tanh u + C$$

$$\frac{d}{dx}[\coth u] = -\operatorname{csch}^2 u \frac{du}{dx} \qquad \int \operatorname{csch}^2 u \, du = -\coth u + C$$

$$\frac{d}{dx}[\operatorname{sech} u] = -\operatorname{sech} u \tanh u \frac{du}{dx} \qquad \int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$$

$$\frac{d}{dx}[\operatorname{csch} u] = -\operatorname{csch} u \coth u \frac{du}{dx} \qquad \int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$$

7.2.2 THEOREM (*Comparison of Exponential and Logarithmic Functions*). *If* $b > 0$ *and* $b \neq 1$, *then:*

$b^0 = 1$	$\log_b 1 = 0$
$b^1 = b$	$\log_b b = 1$
range $b^x = (0, +\infty)$	domain $\log_b x = (0, +\infty)$
domain $b^x = (-\infty, +\infty)$	range $\log_b x = (-\infty, +\infty)$
$y = b^x$ *is continuous on* $(-\infty, +\infty)$	$y = \log_b x$ *is continuous on* $(0, +\infty)$

7.2.3 THEOREM (*Algebraic Properties of Logarithms*). *If $b > 0$, $b \neq 1$, $a > 0$, $c > 0$, and r is any real number, then:*

$\log_b(ac) = \log_b a + \log_b c$	Product property
$\log_b(a/c) = \log_b a - \log_b c$	Quotient property
$\log_b(a^r) = r \log_b a$	Power property
$\log_b(1/c) = -\log_b c$	Reciprocal property

$$\frac{d}{dx}[\cos^{-1} u] = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad (-1 < u < 1)$$

$$\frac{d}{dx}[\tan^{-1} u] = \frac{1}{1+u^2}\frac{du}{dx} \qquad (-\infty < u < +\infty)$$

$$\frac{d}{dx}[\sec^{-1} u] = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad (1 < |u|)$$

needed are

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$$

$$\int \frac{du}{1+u^2} = \tan^{-1} u + C$$

$$\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} |u| + C$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \left|\frac{u}{a}\right| + C$$

formula as

$$\frac{d}{dx}[\log_b x] = \frac{1}{x \ln b}, \quad x > 0$$

$$\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0$$

$$\frac{d}{dx}[\log_b u] = \frac{1}{u \ln b} \cdot \frac{du}{dx}$$

and

$$\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx}$$

$$\int \frac{1}{u}\,du = \ln|u| + C$$

$$\frac{d}{dx}[b^x] = b^x \ln b \tag{12}$$

In the special case where $b = e$ we have $\ln e = 1$, so that (12) becomes

$$\frac{d}{dx}[e^x] = e^x \tag{13}$$

Moreover, if u is a differentiable function of x, then it follows from (12) and (13) that

$$\frac{d}{dx}[b^u] = b^u \ln b \cdot \frac{du}{dx} \quad \text{and} \quad \frac{d}{dx}[e^u] = e^u \cdot \frac{du}{dx} \tag{14–15}$$

$$\int b^u \, du = \frac{b^u}{\ln b} + C \qquad \text{and} \qquad \int e^u \, du = e^u + C$$